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ON THE DETERMINATION OF THE VIEW FUNCTION TO THE IMAGES OF A SURFACE IN A NONPLANAR SPECULAR REFLECTOR*

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Abstract—General mathematical expressions are presented for (1) the view factor to only the primary image of one surface as seen by another surface in a nonplanar specular reflector, and (2) the view factor to secondary images in a multiple imaging situation involving nonplanar specular reflectors. Vector notation is used to present the general formulations that are independent of a specific coordinate system. The resulting expressions are conceptually simple and are identifiable with the commonly known expression for the view factor between diffuse surfaces. The formulations extend the application of the "method of images" to enclosures containing specular surfaces of arbitrary topography. Two examples are given which illustrate the use of the general expressions.

NOMENCLATURE‡

D,	length of raypath between source and receiver;
dA,	incremental area on source or receiver surfaces;
dF,	view factor between two incremental areas;
f^C,	triple product vector equations;
f^D,	dot product vector equations;
f^S,	function defining surfaces of enclosure (Part B);
H,	function defining surface of source, reflector, or receiver (Part A);
I,	radiant flux;
M,	total number of patterns;
m,	maximum size of n;
$\mathbf{N}$,	vector denoting surface normal;
n,	total number of reflections in a pattern;
P,	surface points defining the vectors $\mathbf{V}$;
p,	number of reflectors in enclosure;
$\mathbf{V}$,	vector representing length and direction of a ray;
θ,	angle between ray and source or receiver normals;
ρ,	reflectance;
ϕ,	incident or reflection angle on reflectors.

Subscripts, Part A

1,	receiver surface;
2,	source surface;
2′,	image of source;
$\imath$,	angle of incidence $\phi_\imath$;
r,	reflector;
$\imath$,	angle of reflection $\phi_\imath$.

Subscripts, Part B

S_i,	reflector identification;
S_{n+1},	receiver identification;
S_0,	source identification.

Superscripts, Part B

i,	identification of reflection in pattern;
α,	pattern identification.

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‡ Applies only to Parts A and B of Section 2. Nomenclature in examples (Section 3) is self-explanatory.

1. INTRODUCTION

THE DETERMINATION of view functions, and consequently, view factors, is essential in the calculation of radiant heat transfer between surfaces. In simple calculations, a numerical value representing that portion of the radiant energy leaving one surface and incident upon another is called the view factor (or the angle factor or configuration factor). For complex calculations where local radiant heat-transfer variations are taken into account, a function representing that portion of radiant flux leaving one elemental area and incident upon another is called a view function. The view factor results from the integration of a view function over the areas of the surfaces involved.

Because the mathematical formulations for real surface radiant heat exchange are complex and the subsequent solutions are difficult, a number of simplifying idealizations are generally necessary. These idealizations are related to the nature of the interaction of thermal radiation with a surface and usually pertain to (1) the spatial distributions of reflected and emitted radiation in the half-space above a surface, and (2) the dependence of surface properties on wavelength. In radiative analysis two alternative idealizations prevail regarding the spatial distributions of emitted and reflected radiation: a surface is idealized as either diffuse (Lambertian) or specular (mirror-like). With the diffuse idealization, the spatial distributions of both emitted and reflected radiation are Lambertian. For the specular case, emitted radiation is also Lambertian in distribution; however, the spatial distribution of reflected radiation obeys the "law of reflection" for optically smooth surfaces.

Until recently, analyses have been based almost exclusively upon the diffuse idealization, due to the fact that no conceptually simple analytical technique had been formulated for applying the specular idealization. In spacecraft technology where radiant heat transfer plays a dominant role, it has now been recognized that radiant heat transfer between surfaces of a specular character is perhaps the more common situation, and a number of papers dealing with the specular idealization have appeared in the literature [1–6]. These papers describe analytical techniques for handling the more computationally difficult specular idealization, as well as comparative data demonstrating that important differences can occur when computations are based on one or the other of the two idealizations. Papers published to date deal only with planar and axially symmetrical enclosures involving just the internal surfaces of cylinders and cones.

Among the analytical techniques for handling the specularly idealized situation is the "method of images" [1, 2]. The imaging technique considers the radiation reflected from a specular surface as having originated from the image of the source of that radiation, with the imaged surface appearing as a diffuse source of radiation. Thus, calculations used for the radiant heat-transfer within an enclosure containing specular reflectors are similar to those applied to entirely diffuse enclosures. Therefore the contribution of specularly reflected radiation to the radiant heat transfer can be accounted for by determining the view factors to the images formed in the specular reflectors of the enclosure. Since the images are treated as diffuse sources of radiation, the usual expression for the view factor applies, but the expression must be multiplied by the reflectance of the reflector to account for attenuation in intensity resulting from absorptance by the reflector. The resulting general expression for the view factor to an image in a specular reflector is (Fig. 1):

$$\mathrm{d}A_1\,\mathrm{d}F_{1-2'} = \rho_r \frac{\cos\theta_1 \cos\theta_2'}{\pi D_{1-2'}^2}\,\mathrm{d}A_1\,\mathrm{d}A_2', \quad (1)$$

where $\mathrm{d}F_{1-2'}$ is the incremental view factor from $\mathrm{d}A_1$ to the image of $\mathrm{d}A_2$; $\mathrm{d}A_1$ is an incremental area on surface 1; $\mathrm{d}A_2'$ is an incremental area on the image of surface 2 (the prime denotes image) corresponding to a homologous incremental area $\mathrm{d}A_2$ on surface 2; $D_{1-2'}$ is the distance between $\mathrm{d}A_1$ and $\mathrm{d}A_2'$; θ_1 is the angle formed by the normal at $\mathrm{d}A_1$ and the vector from $\mathrm{d}A_1$ to

dA'_2; θ'_2 is the angle formed by the normal at dA'_2 and the vector from dA'_2 to dA_1; and ρ_r is the reflectance of the reflector.

When the specular reflector is a planar surface, the application of equation (1) is straightforward,

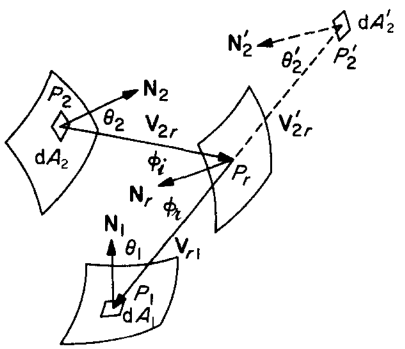


FIG. 1. An arbitrary arrangement of three surfaces, one of which is a specular reflector.

since the location, size, and shape of the image can readily be obtained. However, when the specular reflector is nonplanar, these parameters are more difficult to determine. The use of simple geometric optics in the non-planar case would require ray-tracing to locate and to thoroughly describe an image. Since this approach would be exceedingly difficult and time-consuming, it is the purpose of this paper to present a general analytical technique for determining the view factor to an image in a nonplanar specular reflector.

The derivation of general expressions is presented in two parts: (1) the formulation for the view factor to only the primary image of one surface as seen by another in a specular reflector; and (2), the formulation for the view factor to any secondary image that could be formed by multiple reflection. In addition, a sample problem involving only a single reflection, and thus only a primary image, is presented, and numerical results are given. Finally, a specific problem involving multiple reflection is included to illustrate the application of the general expressions to secondary images.

2. DERIVATION OF GENERAL EXPRESSIONS

As was pointed out by Lin and Sparrow [7], an essential step in the analytical formulation of expressions for the view factor to an image in a nonplanar reflector is the determination of the relationships between the points of origin, reflection, and final incidence of a ray leaving a source surface, being specularly reflected, and finally arriving at a receiver surface. To achieve this, a vector representation of a typical reflection pattern within an enclosure is used here. The use of vector analysis allows the laws of reflection for specular surfaces to be represented in a general form independent of the coordinate system of the enclosure, thus leading to a conceptually simple general formulation.

A. View factor to the primary image

The simplest case involving the reflection of radiation from a nonplanar specular surface is that for which only a single reflection is considered; thus, only the view factor to the primary image is sought. For such a situation, consider an arbitrary arrangement of three surfaces, one of which is a specular reflector, as shown in Fig. 1.

In Fig. 1, a ray emitted from dA_2 at point P_2, striking the reflector at point P_r, and incident upon dA_1 at point P_1, will, as seen from P_1, appear to have originated at a point P'_2, at a distance $|\mathbf{V}'_{2r}|$ in back of the reflector. Point P'_2 is the image of point P_2. Therefore, if one considers an elemental area dA'_2, located at P'_2, whose normal is θ'_2, relative to $\mathbf{V}'_{2r}$, then the view function from dA_1 to dA'_2 is given by equation (1). If dA'_2 is the image of dA_2, then as $dA_2 \to 0$, the principles of geometric optics for planar surfaces apply to these incremental areas, so that:

$$\cos\theta'_2 \equiv \cos\theta_2,$$

$$|\mathbf{V}'_{2r}| \equiv |\mathbf{V}_{2r}|, \tag{2}$$

$$dA'_2 \equiv dA_2.$$

Thus, the view function to the primary image of

A_2 becomes

$$dA_1\, dF_{1-2'} = \rho_r \frac{\cos\theta_1 \cos\theta_2}{\pi(|\mathbf{V}_{r1}| + |\mathbf{V}_{2r}|)^2}\, dA_1\, dA_2, \quad (3)$$

where $D_{1-2'}$ in equation (1) has been replaced by the sum of $|\mathbf{V}_{2r}|$ and $|\mathbf{V}_{r1}|$. The cosine terms appearing in equation (3) can be obtained from the following vector dot products:

$$\left.\begin{aligned} \cos\theta_1 &= \frac{(-\mathbf{V}_{r1})\cdot(\mathbf{N}_1)}{|\mathbf{V}_{r1}|\,|\mathbf{N}_1|} \quad \text{and} \\ \cos\theta_2 &= \frac{(\mathbf{V}_{2r})\cdot(\mathbf{N}_2)}{|\mathbf{V}_{2r}|\,|\mathbf{N}_2|}. \end{aligned}\right\} \quad (4)$$

Furthermore, if surfaces 1 and 2 are defined by the equations $H_1(\xi_1, \xi_2, \xi_3) = 0$ and $H_2(\xi_1, \xi_2, \xi_3) = 0$, respectively, then:

$$\left.\begin{aligned} \mathbf{N}_1 &= \text{Grad}\, H_1 = \nabla H_1, \\ \mathbf{N}_2 &= \text{Grad}\, H_2 = \nabla H_2, \\ |\mathbf{N}_1| &= |\nabla H_1|, \\ \text{and} \quad |\mathbf{N}_2| &= |\nabla H_2|. \end{aligned}\right\} \quad (5)$$

Since the vectors $\mathbf{V}_{2r}$ and $\mathbf{V}_{r1}$ are defined by the reflection point P_r, the evaluation of the view function to the image of dA_2 requires a knowledge of the relationship between the coordinates of P_r and the points P_1 and P_2. One constraint on the coordinates of P_r is that P_r must be on the surface of the reflector which is defined by the equation $H_3(\xi_1, \xi_2, \xi_3) = 0$. The remaining constraints are obtained from the "law of reflection" [8] which requires that:

1. the incident angle ϕ_i must equal the reflection angle ϕ_r; or $\cos\phi_i = \cos\phi_r$, so

$$\frac{(\mathbf{V}_{2r})\cdot(\mathbf{N}_r)}{|\mathbf{V}_{2r}|\,|\mathbf{N}_r|} + \frac{(\mathbf{V}_{r1})\cdot(\mathbf{N}_r)}{|\mathbf{V}_{r1}|\,|\mathbf{N}_r|} = 0; \quad (6)$$

2. the incident ray, reflected ray, and normal at P_r must be coplanar, so that

$$\mathbf{N}_r \cdot (\mathbf{V}_{2r} \times \mathbf{V}_{r1}) = 0. \quad (7)$$

Note that equation (3) can be rewritten in the vector form using appropriate substitutions of equations (4) and (5).

The view factor to the image of surface 2 as seen from surface 1 is obtained by integrating equation (3) over the appropriate limits. The limits of integration over surfaces 1 and 2 are conditional. Two general conditions, or their combination, that might occur are:

1. The image of surface 2 is not completely contained, as viewed from surface 1, within the boundaries of the reflector; in this case, the integration is bounded by the physical boundaries of the reflector, and thus the integration is over that portion of A_2 that A_1 can see as imaged.
2. The image of surface 2 as seen from surface 1 is completely contained within the boundaries of the reflector; in this case, the integration is over the entire areas of surfaces 1 and 2.

B. View factor to secondary images

The development thus far is useful only in determining the view factor to primary images, i.e. images that result from a single specular reflection between the receiver and the source. However, for an enclosure having one or more specular surfaces of such geometry or so arranged that many specular reflections are possible in the transmission of radiant energy from the source to the receiver, a means for determining view functions, and thus view factors, to secondary images is required.

The principal difficulty in developing a general procedure to determine view functions to secondary images is that of accounting—both systematically and mathematically—for all interreflections and resulting images. If the development is limited to enclosures containing a finite number of specular surfaces p, each of which is defined by a continuous function, then each specular surface can be uniquely identified by a number from the set 1, 2, 3, . . ., p. A particular ordering of numbers from the set 1, 2, 3, . . ., p, designating the sequential ordering of reflections among the specular surfaces in the transmission of radiation from the source to the receiver would result in one or more *particular* images of the source. This sequential ordering of numbers from the set 1,2,3,. . ., p will be referred to as a pattern. For example, the following sequential ordering of numbers 1, 8, 6, 3 would define a pattern of specular reflections, in chronological order, from the source to specular surfaces 1, 8, 6, and 3 and then to the receiver. Since the sequential ordering of reflections from the finite set 1, 2, 3, . . ., p can involve repetitious use of any or all numbers from the set, the number of reflections can be infinite; thus, the possible number of patterns can be infinite. Since the number of possible patterns is large, a number from another set of numbers 1, 2, . . ., ∞ will in turn be used for the identification of a particular pattern; i.e. a number from the second set will denote a particular sequential ordering of numbers from the set 1, 2, 3, . . ., p.

The symbol α will be used as a general designation of the identification number for a pattern; i.e. α is a general designation of a number in the set 1, 2, . . ., ∞. In a particular pattern, the number of reflections between the source and the receiver will be denoted by n where $n = 1, 2, \ldots, \infty$. Any particular reflection will be denoted by i where $i = 1, 2, \ldots, n$. Associated with the ith reflection is a specular surface identification number from the set 1, 2, 3, . . ., p. This surface identification number will be symbolically represented by S_i.

Using the latter symbols, any ray reflected between two specular surfaces and the coordinates of any reflection point in a pattern can be accounted for by the use of a double subscript and superscript notation. Thus the vector $\mathbf{V}^{\alpha,i}_{S_{i-1},S_i}$ represents a ray in the αth pattern occurring at the ith reflection in the chronology of the pattern. The origin of the vector occurs on the specular reflector identified by S_{i-1} and its terminus by S_i. A similar notation for the reflection points defining this vector and the coordinates of the reflection points can be used. The terminus point of the vector would be $P^{\alpha,i}_{S_i}$, or in terms of generated coordinates, this point is $(\xi^{\alpha,i}_{1S_i}, \xi^{\alpha,i}_{2S_i}, \xi^{\alpha,i}_{3S_i})$ while the origin of the vector is $P^{\alpha,i-1}_{S_{i-1}}$ or $(\xi^{\alpha,i-1}_{1,S_{i-1}}, \xi^{\alpha,i-1}_{2,S_{i-1}}, \xi^{\alpha,i-1}_{3,S_{i-1}})$. The symbols S_0 and S_{n+1} will denote the source and receiver surfaces, respectively.

In general, for a pair of source and receiver surfaces, the maximum number M of patterns for an arbitrary enclosure of p number of specular surfaces is given by the expression:

$$M = \sum_{n=1}^{m} (p)^n, \tag{8}$$

where m is the maximum number of reflections which contribute to the radiant heat transfer between the source surface and the receiver surface. Note that m is not necessarily the maximum number of possible reflections but, rather, an imposed arbitrary limit; also, the maximum number of patterns given by equation (8) does not take into account the arrangement or topography of the specular surfaces in the enclosure. Thus, for a specific configuration, certain patterns are likely to be physically impossible because of surface arrangement or topography. For example, if a specular surface is planar or convex, those patterns which require two or more consecutive reflections are physically impossible.

To illustrate the application of the notation introduced thus far, consider an enclosure containing two specularly reflecting surfaces limited to a maximum of four reflections for a pattern. Such an enclosure is illustrated in

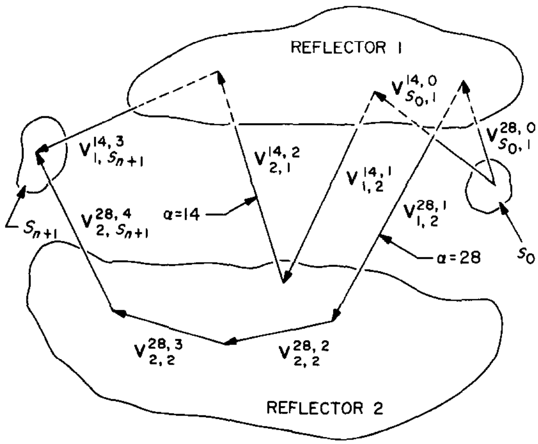


FIG. 2. Diagram illustrating notation for several patterns using a two-specular surface configuration.

Fig. 2, where S_0 and S_{n+1} are arbitrary source and receiver surfaces, respectively. From equation (8), the maximum number of patterns M for $m = 4$ is 30. The number of patterns for $n = 1, 2, 3$, and 4 reflections is shown in Table 1. The sum is, of course, 30. This table clearly shows the correlation between the rapid increase in M with the increase in maximum reflections m. A tabulation of the pattern number α against the sequence of the chronological surfaces in the propagation of the pattern and the associated sequence of vectors is shown in Table 2.

Table 1. Number of patterns vs. number of reflections n in each pattern for two specular surface enclosures

Number of reflections in pattern, n	Number of patterns
1	2
2	4
3	8
4	16
	$30 = M$

The rays shown in Fig. 2 represent the interreflection pattern for patterns 14 and 28 of Table 2.

In an arbitrary enclosure containing one or more specular surfaces, the number of possible patterns can be very large—even infinite. For such situations, a limit on the number of reflections, and thus the number of possible patterns, is necessary for practical reasons. One means of imposing a limit is to consider the reflectances of the specular surfaces involved in a particular pattern. For each specular reflection, the reflected radiant energy would be attenuated in proportion to the reflectance of the surface. The ratio of attenuated radiant energy of the α pattern of n reflections to the initial radiant energy is:

$$I_\alpha/I_0 = \prod_{i=1}^{n} \rho_{S_i}, \tag{9}$$

where I_α/I_0 is the ratio of attenuated to unattenuated radiant energy for the α pattern of n reflections. As an example of this attenuation, if $\rho = 0{\cdot}8$ is the reflectance for all the surfaces involved in a pattern of six reflections, then $I_\alpha/I_0 = 0{\cdot}26$; however, if $\rho = 0{\cdot}5$, then after six reflections, $I_\alpha/I_0 = 0{\cdot}016$. Although this criterion neglects the effects of geometry and surface arrangement in the enclosure, it is possible to obtain a crude estimate of the contribution to the radiant transfer as the number of reflections of the patterns increase.

The total view factor for the transfer of radiative energy between two surfaces of a configuration by multiple specular reflection is equal to the sum of the view factors to the images resulting from all patterns between these two surfaces. Therefore, the general expression for the view function to any secondary image formed by a particular pattern of reflections must be determined.

When the above notation is used, the general mathematical expressions necessary to evaluate the view function to any secondary image take a form similar to that obtained in the primary image development as explained in the preceding section. Rather than performing a rigorous development for the general expression of the view function to secondary images, the expression already developed for the primary image view factor (equation 3) will be used. In this expression, it will be recalled, the two cosine terms in the numerator represent the angles at which a ray leaves a source surface and arrives at the receiver surface. It then follows that these two cosine terms maintain the same significance for both the single and multiple reflection view functions. The term in the denominator of equation (3) represents the square of total path distance traveled by the ray from the source to the receiver surface; likewise, the denominator of the multiple reflection view function represents the total path length squared.

Thus, when considering the secondary image formed by the αth pattern of n reflections, the view functions to this image on surface number S_0

Table 2. Sequence of all patterns for a maximum of four reflections from two specular surfaces

Pattern number	Surface sequence	Vector sequence				
1	$S_0, 1, S_{n+1}$	$\mathbf{V}^{1,0}_{S_0,1}$	$\mathbf{V}^{1,1}_{1,S_{n+1}}$			
2	$S_0, 2, S_{n+1}$	$\mathbf{V}^{2,0}_{S_0,2}$	$\mathbf{V}^{2,1}_{2,S_{n+1}}$			
3	$S_0, 1, 1, S_{n+1}$	$\mathbf{V}^{3,0}_{S_0,1}$	$\mathbf{V}^{3,1}_{1,1}$	$\mathbf{V}^{3,2}_{1,S_{n+1}}$		
4	$S_0, 1, 2, S_{n+1}$	$\mathbf{V}^{4,0}_{S_0,1}$	$\mathbf{V}^{4,1}_{1,2}$	$\mathbf{V}^{4,2}_{2,S_{n+1}}$		
5	$S_0, 2, 1, S_{n+1}$	$\mathbf{V}^{5,0}_{S_0,2}$	$\mathbf{V}^{5,1}_{2,1}$	$\mathbf{V}^{5,2}_{1,S_{n+1}}$		
6	$S_0, 2, 2, S_{n+1}$	$\mathbf{V}^{6,0}_{S_0,2}$	$\mathbf{V}^{6,1}_{2,2}$	$\mathbf{V}^{6,2}_{2,S_{n+1}}$		
7	$S_0, 1, 1, 1, S_{n+1}$	$\mathbf{V}^{7,0}_{S_0,1}$	$\mathbf{V}^{7,1}_{1,1}$	$\mathbf{V}^{7,2}_{1,1}$	$\mathbf{V}^{7,3}_{1,S_{n+1}}$	
8	$S_0, 1, 1, 2, S_{n+1}$	$\mathbf{V}^{8,0}_{S_0,1}$	$\mathbf{V}^{8,1}_{1,1}$	$\mathbf{V}^{8,2}_{1,2}$	$\mathbf{V}^{8,3}_{2,S_{n+1}}$	
9	$S_0, 1, 2, 2, S_{n+1}$	$\mathbf{V}^{9,0}_{S_0,1}$	$\mathbf{V}^{9,1}_{1,2}$	$\mathbf{V}^{9,2}_{2,2}$	$\mathbf{V}^{9,3}_{2,S_{n+1}}$	
10	$S_0, 2, 2, 2, S_{n+1}$	$\mathbf{V}^{10,0}_{S_0,2}$	$\mathbf{V}^{10,1}_{2,2}$	$\mathbf{V}^{10,2}_{2,2}$	$\mathbf{V}^{10,3}_{2,S_{n+1}}$	
11	$S_0, 2, 2, 1, S_{n+1}$	$\mathbf{V}^{11,0}_{S_0,2}$	$\mathbf{V}^{11,1}_{2,2}$	$\mathbf{V}^{11,2}_{2,1}$	$\mathbf{V}^{11,3}_{1,S_{n+1}}$	
12	$S_0, 2, 1, 1, S_{n+1}$	$\mathbf{V}^{12,0}_{S_0,2}$	$\mathbf{V}^{12,1}_{2,1}$	$\mathbf{V}^{12,2}_{1,1}$	$\mathbf{V}^{12,3}_{1,S_{n+1}}$	
13	$S_0, 2, 1, 2, S_{n+1}$	$\mathbf{V}^{13,0}_{S_0,2}$	$\mathbf{V}^{13,1}_{2,1}$	$\mathbf{V}^{13,2}_{1,2}$	$\mathbf{V}^{13,3}_{2,S_{n+1}}$	
14	$S_0, 1, 2, 1, S_{n+1}$	$\mathbf{V}^{14,0}_{S_0,1}$	$\mathbf{V}^{14,1}_{1,2}$	$\mathbf{V}^{14,2}_{2,1}$	$\mathbf{V}^{14,3}_{1,S_{n+1}}$	
15	$S_0, 1, 1, 1, 1, S_{n+1}$	$\mathbf{V}^{15,0}_{S_0,1}$	$\mathbf{V}^{15,1}_{1,1}$	$\mathbf{V}^{15,2}_{1,1}$	$\mathbf{V}^{15,3}_{1,1}$	$\mathbf{V}^{15,4}_{1,S_{n+1}}$
16	$S_0, 1, 1, 2, 1, S_{n+1}$	$\mathbf{V}^{16,0}_{S_0,1}$	$\mathbf{V}^{16,1}_{1,1}$	$\mathbf{V}^{16,2}_{1,2}$	$\mathbf{V}^{16,3}_{2,1}$	$\mathbf{V}^{16,4}_{1,S_{n+1}}$
17	$S_0, 1, 2, 1, 1, S_{n+1}$	$\mathbf{V}^{17,0}_{S_0,1}$	$\mathbf{V}^{17,1}_{1,2}$	$\mathbf{V}^{17,2}_{2,1}$	$\mathbf{V}^{17,3}_{1,1}$	$\mathbf{V}^{17,4}_{1,S_{n+1}}$
18	$S_0, 1, 1, 1, 2, S_{n+1}$	$\mathbf{V}^{18,0}_{S_0,1}$	$\mathbf{V}^{18,1}_{1,1}$	$\mathbf{V}^{18,2}_{1,1}$	$\mathbf{V}^{18,3}_{1,2}$	$\mathbf{V}^{18,4}_{2,S_{n+1}}$
19	$S_0, 2, 1, 1, 1, S_{n+1}$	$\mathbf{V}^{19,0}_{S_0,2}$	$\mathbf{V}^{19,1}_{2,1}$	$\mathbf{V}^{19,2}_{1,1}$	$\mathbf{V}^{19,3}_{1,1}$	$\mathbf{V}^{19,4}_{1,S_{n+1}}$
20	$S_0, 1, 1, 2, 2, S_{n+1}$	$\mathbf{V}^{20,0}_{S_0,1}$	$\mathbf{V}^{20,1}_{1,1}$	$\mathbf{V}^{20,2}_{1,2}$	$\mathbf{V}^{20,3}_{2,2}$	$\mathbf{V}^{20,4}_{2,S_{n+1}}$
21	$S_0, 1, 2, 1, 2, S_{n+1}$	$\mathbf{V}^{21,0}_{S_0,1}$	$\mathbf{V}^{21,1}_{1,2}$	$\mathbf{V}^{21,2}_{2,1}$	$\mathbf{V}^{21,3}_{1,2}$	$\mathbf{V}^{21,4}_{2,S_{n+1}}$
22	$S_0, 2, 1, 1, 2, S_{n+1}$	$\mathbf{V}^{22,0}_{S_0,2}$	$\mathbf{V}^{22,1}_{2,1}$	$\mathbf{V}^{22,2}_{1,1}$	$\mathbf{V}^{22,3}_{1,2}$	$\mathbf{V}^{22,4}_{2,S_{n+1}}$
23	$S_0, 2, 1, 2, 1, S_{n+1}$	$\mathbf{V}^{23,0}_{S_0,2}$	$\mathbf{V}^{23,1}_{2,1}$	$\mathbf{V}^{23,2}_{1,2}$	$\mathbf{V}^{23,3}_{2,1}$	$\mathbf{V}^{23,4}_{1,S_{n+1}}$
24	$S_0, 1, 2, 2, 1, S_{n+1}$	$\mathbf{V}^{24,0}_{S_0,1}$	$\mathbf{V}^{24,1}_{1,2}$	$\mathbf{V}^{24,2}_{2,2}$	$\mathbf{V}^{24,3}_{2,1}$	$\mathbf{V}^{24,4}_{1,S_{n+1}}$
25	$S_0, 2, 1, 2, 2, S_{n+1}$	$\mathbf{V}^{25,0}_{S_0,2}$	$\mathbf{V}^{25,1}_{2,1}$	$\mathbf{V}^{25,2}_{1,2}$	$\mathbf{V}^{25,3}_{2,2}$	$\mathbf{V}^{25,4}_{2,S_{n+1}}$
26	$S_0, 2, 2, 1, 1, S_{n+1}$	$\mathbf{V}^{26,0}_{S_0,2}$	$\mathbf{V}^{26,1}_{2,2}$	$\mathbf{V}^{26,2}_{2,1}$	$\mathbf{V}^{26,3}_{1,1}$	$\mathbf{V}^{26,4}_{1,S_{n+1}}$
27	$S_0, 2, 2, 1, 2, S_{n+1}$	$\mathbf{V}^{27,0}_{S_0,2}$	$\mathbf{V}^{27,1}_{2,2}$	$\mathbf{V}^{27,2}_{2,1}$	$\mathbf{V}^{27,3}_{1,2}$	$\mathbf{V}^{27,4}_{2,S_{n+1}}$
28	$S_0, 1, 2, 2, 2, S_{n+1}$	$\mathbf{V}^{28,0}_{S_0,1}$	$\mathbf{V}^{28,1}_{1,2}$	$\mathbf{V}^{28,2}_{2,2}$	$\mathbf{V}^{28,3}_{2,2}$	$\mathbf{V}^{28,4}_{2,S_{n+1}}$
29	$S_0, 2, 2, 2, 1, S_{n+1}$	$\mathbf{V}^{29,0}_{S_0,2}$	$\mathbf{V}^{29,1}_{2,2}$	$\mathbf{V}^{29,2}_{2,2}$	$\mathbf{V}^{29,3}_{2,1}$	$\mathbf{V}^{29,4}_{1,S_{n+1}}$
30	$S_0, 2, 2, 2, 2, S_{n+1}$	$\mathbf{V}^{30,0}_{S_0,2}$	$\mathbf{V}^{30,1}_{2,2}$	$\mathbf{V}^{30,2}_{2,2}$	$\mathbf{V}^{30,3}_{2,2}$	$\mathbf{V}^{30,4}_{2,S_{n+1}}$

as viewed from surface number S_{n+1} are:

$$dA_{S_{n+1}} dF_{S^{\alpha}_{n+1}, S_0} = \left[\prod_{i=1}^{n} \rho_{S_i}\right] \frac{\cos\theta^{\alpha}_{S_0}\cos\theta^{\alpha}_{S_{n+1}}}{(D^{\alpha})^2} dA_{S_{n+1}}\, dA_{S_0}, \tag{10}$$

where

$$D^{\alpha} = \sum_{i=1}^{n+1} |\mathbf{V}^{\alpha,i}_{S_{i-1},S_i}|,$$

and

$$\cos\theta^{\alpha}_{S_0} = \frac{(\mathbf{N}^{\alpha,0}_{S_0})\cdot(\mathbf{V}^{\alpha,1}_{S_0,S_1})}{|\mathbf{N}^{\alpha,0}_{S_0}|\cdot|\mathbf{V}^{\alpha,1}_{S_0,S_1}|};$$

$$\cos\theta_{S_{n+1}} = \frac{(\mathbf{N}^{\alpha,n+1}_{S_{n+1}})\cdot(\mathbf{V}^{\alpha,n+1}_{S_n,S_{n+1}})}{|\mathbf{N}^{\alpha,n+1}_{S_{n+1}}|\;|\mathbf{V}^{\alpha,n+1}_{S_n,S_{n+1}}|},$$

and where $(S_0, S_1, S_2, \ldots, S_{n+1})$ is the sequence of surface identification numbers associated with the αth pattern. As in the case of the view function to the primary image, the view factor to a secondary image is obtained by integrating equation (10) over the appropriate limits. The limits of integration as well as the relationships between the variables appearing in the integrand are obtained by solving a system of equations similar to equations (6) and (7). This system of equations for the multiple reflection case specifies the conditions under which the "law of reflection" is satisfied at each reflection point, and yields the relationships between the source point, reflection points, and final incidence point for the secondary image patterns. For the αth pattern which involves n reflections at points $(P_{S_1}^{\alpha,1}, P_{S_2}^{\alpha,2}, P_{S_3}^{\alpha,3}, \ldots, P_{S_n}^{\alpha,n})$ on the sequence of surfaces $(S_1, S_2, S_3, \ldots, S_i, \ldots, S_n)$, where the S_i's are chosen from the set of surface identification numbers, the coordinates of the reflection points are related to the source point $P_{S_0}^{\alpha,0}$ on surface S_0, and the final incidence point $P_{S_{n+1}}^{\alpha,n+1}$ on surface S_{n+1}, by the system of $(3n + 2)$ simultaneous equations of the form:

$$\frac{(\mathbf{V}_{S_{i-1},S_i}^{\alpha,i})\cdot(\mathbf{N}_{S_i}^{\alpha,i})}{|\mathbf{V}_{S_{i-1},S_i}^{\alpha,i}|\,|\mathbf{N}_{S_i}^{\alpha,i}|} + \frac{(\mathbf{N}_{S_i}^{\alpha,i})\cdot(\mathbf{V}_{S_i,S_{i+1}}^{\alpha,i+1})}{|\mathbf{N}_{S_i}^{\alpha,i}|\,|\mathbf{V}_{S_i,S_{i+1}}^{\alpha,i+1}|}$$

$$= f_{S_{i-1},S_i,S_{i+1}}^{D}(P_{S_{i-1}}^{\alpha,i-1}, P_{S_i}^{\alpha,i}, P_{S_{i+1}}^{\alpha,i+1}) = 0, \quad (11)$$

$$\mathbf{N}_{S_i}^{\alpha,i}\cdot(\mathbf{V}_{S_{i-1},S_i}^{\alpha,i} \times \mathbf{V}_{S_i,S_{i+1}}^{\alpha,i+1}) =$$

$$f_{S_{i-1},S_i,S_{i+1}}^{C}(P_{S_{i-1}}^{\alpha,i-1}, P_{S_i}^{\alpha,i}, P_{S_{i+1}}^{\alpha,i+1}) = 0, \quad (12)$$

$$f_{S_i}^{F}(P_{S_i}^{\alpha,i}) = 0, \quad (13)$$

for $i = 1, 2, 3, \ldots, n$,

as well as

$$f_{S_0}^{F}(P_{S_0}^{\alpha,0}) = 0, \quad (14)$$

and

$$f_{S_{n+1}}^{F}(P_{S_{n+1}}^{\alpha,0+1}) = 0. \quad (15)$$

The functions $f_{S_i}^F$ for $i = 1, 2, 3, \ldots, n$ are the equations of the specular surfaces involved in the pattern. The $f_{S_0}^F$ and $f_{S_{n+1}}^F$ are the equations of the source and the receiver surfaces, respectively. Each of the functions $f_{S_{i-1},S_1,S_{i+1}}^D$, which result for the expansion of the dot product vector equations, in the general case relate the nine coordinates of three successive reflection points so as to insure that the angles of incidence and reflection are equal at each reflection point. Likewise, the functions $f_{S_{i-1},S_i,S_{i+1}}^C$, which result from the expansion of the triple-product vector equations, insure the coplanarity of the incident ray, the reflected ray and the normal to the surface at the reflection point.

The $3n + 2$ equations in general will contain three $(n + 2)$ coordinates of the pattern under investigation. Thus, if four coordinates of the pattern are specified, the above system of equations will yield a solution, provided the pattern is physically capable of satisfying the constraints imposed by these coordinates. Usually, two of the specified coordinates would be related to the source point of the pattern and the other two would be related to either the receiver point of the pattern or the first reflection point of the pattern.

Careful examination of the multiple reflection formulation in terms of equations (11) and (12) reveals that the number of functional forms of f^D and f^C required to represent reflections that make up all possible patterns is related only to the number of surfaces in a configuration. This maximum number of functional forms of f^D and f^C required to describe all possible patterns involving at least one specular reflection for a configuration of p_D diffuse and p_S specular surfaces is $(p_S + p_D)^2$. Due to the reciprocity theorem [8] as well as the physical impossibility of some reflections, there will be considerably fewer functional forms in most applications. For example, the functional forms for a (1, 2, 3) reflection and a (3, 2, 1) reflection are identical; also, when considering reciprocity, the number of functional forms is reduced to:

$$(p_S)\left(\sum_{p=1}^{p_D+p_S}\right)p\ .$$

3. EXAMPLES

A. Example 1—view factor to a primary image

The following example demonstrates the use of the equations derived for a single or primary image. For this purpose, consider a segment of a circumferentially finned cylinder with the non-planar circular wall reflecting radiation specularly, as shown in Fig. 3. For this case, the view factor to the image of the right fin in the cylinder's wall, as seen from the left fin, is desired.

Because of axial symmetry, an elemental radial strip $(b - a)\,d\alpha_1$ at any angle α_1 on the left fin views exactly the same geometrical image of the right fin; thus, dA_1 can be fixed to lie along the p-axis at $\alpha_1 = 0$. A point defining the location of dA_1 is:

$$P_1 = 0, p, 0, \tag{16a}$$

where p is, in cylindrical coordinates, the radial distance from the axis. With P_1 fixed along the p-axis at $\alpha_1 = 0$, the points P_2 and P_r become:

$$P_2 = q \sin\alpha_2, \quad q\cos\alpha_2, L, \tag{16b}$$

$$P_r = a\sin\omega, \quad a\cos\omega, z, \tag{16c}$$

where q and α_2 are the radial and azimuthal coordinates of the right fin, and z and ω are the axial and azimuthal coordinates of the reflection point on the wall of the cylinder. Using these points, the vectors $\mathbf{V}_{1r}$ and $\mathbf{V}_{r2}$ become:

$$\mathbf{V}_{r1} = P_1 - P_r = \mathbf{i}(-a\sin\omega) + \mathbf{j}(-a\cos\omega + p) + \mathbf{k}(-z), \tag{17a}$$

$$\mathbf{V}_{r2} = P_2 - P_r = \mathbf{i}(-q\sin\alpha_2 + \alpha\sin\omega), + \mathbf{j}(q\cos\alpha_2 + a\cos\omega) + \mathbf{k}(-L + z), \tag{17b}$$

with their absolute magnitudes given by:

$$|\mathbf{V}_{1r}| = |[a^2 + p^2 - 2ap\cos\omega + z^2]^{\frac{1}{2}}|, \tag{18a}$$

$$|\mathbf{V}_{r2}| = |[a^2 + q^2 - 2aq\cos(\alpha_2 - \omega) + (L - z)^2]^{\frac{1}{2}}|. \tag{18b}$$

The equations for the planes containing the surfaces of the fin are given by $H_1(\xi_1, \xi_2, \xi_3) = z_1 = 0$ and $H_2(\xi_1, \xi_2, \xi_3) = L - z_2 = 0$; thus, the normal vectors $\mathbf{N}_1$ and $\mathbf{N}_2$ are:

$$\mathbf{N}_1 = +\mathbf{k} \quad \text{and} \quad \mathbf{N}_2 = -\mathbf{k}, \tag{19}$$

with their absolute magnitudes being:

$$|\mathbf{N}_1| = 1 \quad \text{and} \quad |\mathbf{N}_2| = 1. \tag{20}$$

The equation for the surface of the cylinder at any location z, is $H_r(\xi_1, \xi_2, \xi_3) = \xi_1^2 + \xi_2^2 - a^2 = 0$ where $\xi_1 = a\sin\omega$ and $\xi_2 = a\cos\omega$; thus the normal vector at the point of reflection is:

$$\mathbf{N}_1 = \mathbf{i}2a\sin\omega + \mathbf{j}2a\cos\omega, \tag{21}$$

and its absolute magnitude is:

$$|\mathbf{N}_1| = 2a. \tag{22}$$

For this particular example, the limits of integration are terminated over the azimuthal angle α_2, when $(\mathbf{V}_{1r})\cdot(\mathbf{N}_r) = 0$ and

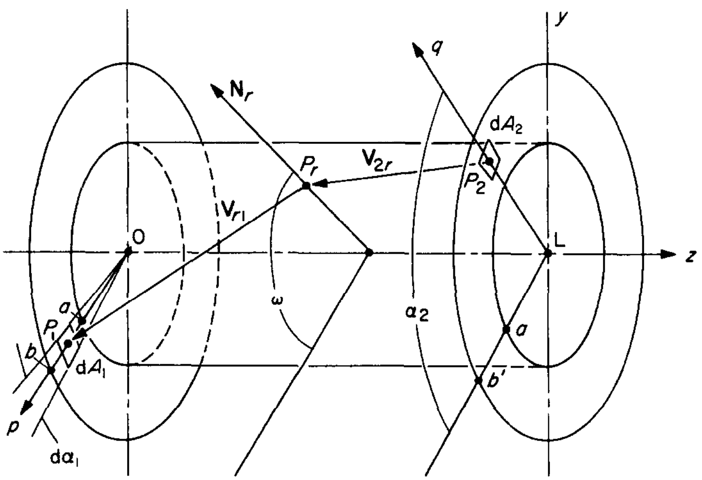


FIG. 3. Cylindrically finned cylinder.

$(\mathbf{V}_{r2}) \cdot (\mathbf{N}_r) = 0$; thus the limits over α_2 are:

$$\alpha_2 = \pm \left(\cos^{-1}\frac{a}{p} + \cos^{-1}\frac{a}{q}\right). \qquad (23)$$

If both fins are of equal outer radius, then the limits of integration over p and q are from a to b. The limit of integration over α_1 is simply from 0 to 2π.

Nondimensionalize the equations by letting $p = \zeta a$, $q = \eta a$, $z = \beta b$, $b = Ma$, and $L = Na$. Now substituting equations (17–23) into equations (3), (6), and (7) and integrating equation (3) over α_2 yields the following set of equations for the view factor to the image:

$$\begin{aligned} F_{1-2} = {} & \frac{2\rho}{\pi(M^2-1)} \\ & \times \int_{\zeta=1}^{M} \int_{\eta=1}^{M} \int_{-\cos^{-1}(1/\eta)-\cos^{-1}(1/\zeta)}^{+\cos^{-1}(1/\eta)+\cos^{-1}(1/\zeta)} \beta\zeta\eta(N-\beta) \\ & \times [1+\zeta^2-2\zeta\cos\omega+\beta^2]^{-\frac{1}{2}} \\ & \times [1+\eta^2-2\eta\cos(\alpha_2-\omega)+(N-\beta)^2]^{-\frac{1}{2}} \\ & \times \{[1+\zeta^2-2\zeta\cos\omega+\beta^2]^{\frac{1}{2}} \\ & + [1+\eta^2-2\eta\cos(\alpha_2-\omega) \qquad (24a) \\ & \qquad + (N-\beta)^2]^{\frac{1}{2}}\}^{-2}\, d\alpha_2\, d\eta\, d\zeta \end{aligned}$$

$$\beta = \zeta N \sin\omega / [\zeta \sin\omega + \eta \sin(\alpha_2 - \omega)] \qquad (24b)$$

$$\frac{\eta\cos(\alpha_2-\omega)-1}{[1+\eta^2-2\eta\cos(\alpha_2-\omega)+(N-\beta)^2]^{\frac{1}{2}}} = \frac{1-\zeta\cos\omega}{[1+\zeta^2-2\zeta\cos\omega+\beta^2]^{\frac{1}{2}}}. \qquad (24c)$$

These equations were programmed for and solved numerically on an IBM 7094 computer. Simultaneous solutions were first obtained for equations (24b) and (24c) for the coordinates of the reflection point on the cylinder wall for systematic values of the coordinates of the source point P_2 and the receiver point P_1. With the reflection point tabulated against a pair of source and receiver point coordinates, numerical solutions were then obtained for equation (24a). The results of these calculations are plotted in Fig. 4.

B. Example 2—view factor for secondary images

To briefly illustrate the technique for formulating the system of equations which describe the patterns for a multiple reflection configuration, consider an infinite channel consisting of a concave specular surface, a convex specular surface, and two parallel diffuse surfaces, as shown in Fig. 5. Only the patterns which lie in the plane of the cross section and originate at a point y_0 on surface 0 and are received at point y_3 on surface 3 are considered.

Because the patterns of this example are now confined to a plane, the f^C's may be ignored. The vectors required to describe the patterns for this configuration are

$$\mathbf{V}_{0,2}^{\alpha,1} = \varepsilon_x[x_2^{\alpha,1} - 1] + \varepsilon_y[(x_2^{\alpha,1})^2 + y_0^{\alpha,0} - 1]$$

$$\mathbf{V}_{0,1}^{\alpha,1} = \varepsilon_x[x_2^{\alpha,1} - 1] + \varepsilon_y[(x_2^{\alpha,1})^2 + y_0^{\alpha,0} - 1]$$

$$\mathbf{V}_{2,3}^{\alpha,i} = \varepsilon_x[x_1^{\alpha,1} - 1] + \varepsilon_y[(x_1^{\alpha,1})^2 + y_0^{\alpha,0}]$$

$$\mathbf{V}_{1,3}^{\alpha,i} = \varepsilon_x[-1 - x_1^{\alpha,i-1}] + \varepsilon_y[y_3^{\alpha,i} - (x_1^{\alpha,i-1})^2]$$

$$\mathbf{V}_{2,1}^{\alpha,i} = \varepsilon_x[x_1^{\alpha,i} - x_2^{\alpha,i-1}] + \varepsilon_y[(x_1^{\alpha,i})^2 - (x_2^{\alpha,i-1})^2 + 1]$$

$$\mathbf{V}_{1,2}^{\alpha,i} = \varepsilon_x[x_2^{\alpha,i} - x_1^{\alpha,i-1}] + \varepsilon_y[(x_2^{\alpha,1})^2 - (x_1^{\alpha,i-1})^2 - 1]$$

$$\mathbf{V}_{2,2}^{\alpha,i} = \varepsilon_x[x_2^{\alpha,i} - x_2^{\alpha,i-1}] + \varepsilon_y[(x_2^{\alpha,i})^2 - (x_2^{\alpha,i-1})^2]$$

$$\mathbf{N}_1^{\alpha,i} = \varepsilon_x[2x_1^{\alpha,i}] + \varepsilon_y[-1]$$

$$\mathbf{N}_2^{\alpha,i} = \varepsilon_x[-2x_2^{\alpha,i}] + \varepsilon_y[+1]$$

where ε_x and ε_y are unit vectors.

By the relationship given above, the maximum number of f^D forms required to describe all patterns is 20. However, pattern (0, 1, 3) and patterns which result in direct reflections back to the diffuse surface such as (0, 1, 0), (0, 2, 0), (3, 1, 3), and (3, 2, 3) have been disregarded. In addition, the patterns which require a ray to pass between two points on surface 1, such as (0, 1, 1), (1, 1, 1), (1, 1, 3), and (1, 1, 2), are physically impossible. Thus, only eleven functional forms are required.

Substitution of these vectors for those of

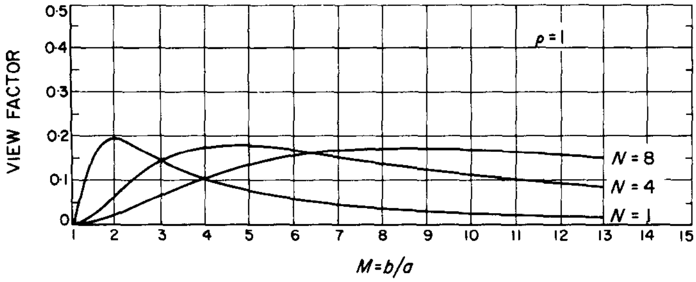


FIG. 4. View factor to the image of a cylindrical fin.

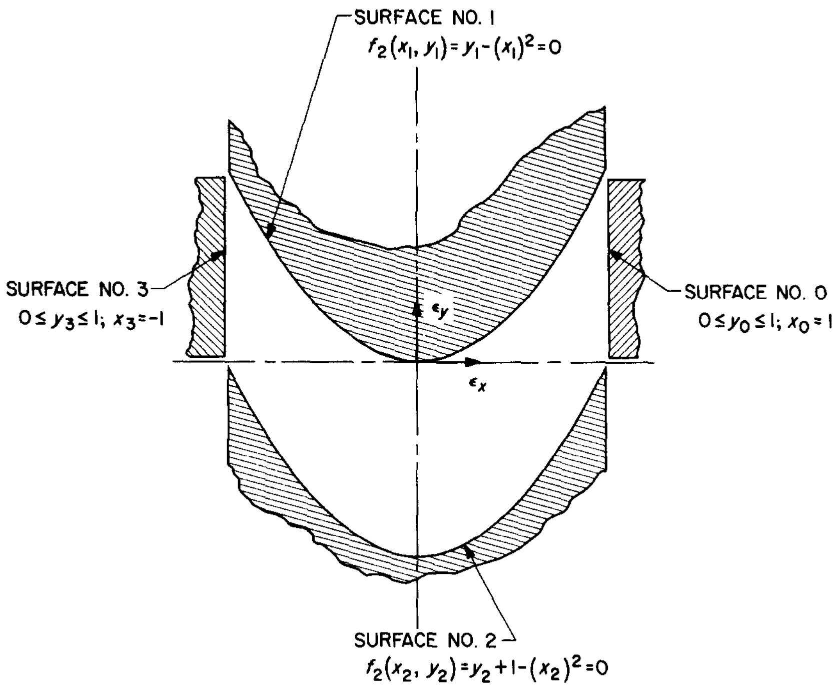


FIG. 5. Convex–concave channel.

equation (11) and expansion yields the functional relationships required to describe all multiple reflection patterns considered. These are:

$$\frac{\mathbf{N}_1^{\alpha,i}\cdot\mathbf{V}_{2,1}^{\alpha,i}}{|\mathbf{N}_1^{\alpha,i}|\,|\mathbf{V}_{2,1}^{\alpha,i}|}+\frac{\mathbf{N}_1^{\alpha,i}\cdot\mathbf{V}_{1,2}^{\alpha,i-1}}{|\mathbf{N}_1^{\alpha,i}|\,|\mathbf{V}_{1,2}^{\alpha,i-1}|}=f^D_{2,1,2}\,(x_1^{\alpha,i-1}, x_1^{\alpha,i}, x_2^{\alpha,i-1})=f^D_{2,1,2}\,(a,b,c)$$

$$=\left\{\frac{a^2-2ab+b^2-1}{[(b-a)^2+(b^2-a^2+1)^2]^{\frac{1}{2}}}+\frac{2bc-b^2-c^2+1}{[(c-b)^2+(c^2-b^2-1)^2]^{\frac{1}{2}}}\right\}(1+4b^2)^{-\frac{1}{2}},$$

$$\frac{\mathbf{N}_2^{\alpha,i}\cdot\mathbf{V}_{1,2}^{\alpha,i}}{|\mathbf{N}_2^{\alpha,i}|\,|\mathbf{V}_{1,2}^{\alpha,i}|}+\frac{\mathbf{N}_2^{\alpha,i}\cdot\mathbf{V}_{2,1}^{\alpha,i+1}}{|\mathbf{N}_2^{\alpha,i}|\,|\mathbf{V}_{2,1}^{\alpha,i+1}|}=f_{1,2,1}^{D}(x_1^{\alpha,i-1},x_2^{\alpha,i},x_1^{\alpha,i+1})=f_{1,2,1}^{D}$$

$$=\left\{\frac{2ab-b^2-a^2-1}{[(b-a)^2+(b^2-a^2-1)^2]^{\frac{1}{2}}}+\frac{b^2-2bc+c^2+1}{[(c-b)^2+(c^2-b^2+1)^2]^{\frac{1}{2}}}\right\}(1+4b^2)^{-\frac{1}{2}},$$

$$\frac{\mathbf{N}_2^{\alpha,i}\cdot\mathbf{V}_{1,2}^{\alpha,i}}{|\mathbf{N}_2^{\alpha,i}|\,|\mathbf{V}_{1,2}^{\alpha,i}|}+\frac{\mathbf{N}_2^{\alpha,i}\cdot\mathbf{V}_{2,2}^{\alpha,i+1}}{|\mathbf{N}_2^{\alpha,i}|\,|\mathbf{V}_{2,2}^{\alpha,i+1}|}=f_{1,2,2}^{D}(x_1^{\alpha,i-1},x_2^{\alpha,i},x_2^{\alpha,i+1})=f_{1,2,2}^{D}(a,b,c)$$

$$=\left\{\frac{2ab-b^2-a^2-1}{[(b-a)^2+(b^2-a^2-1)^2]^{\frac{1}{2}}}+\frac{b^2-2bc+c^2}{[(c-b)^2+(c^2-b^2)^2]^{\frac{1}{2}}}\right\}(1+4b^2)^{-\frac{1}{2}},$$

$$\frac{\mathbf{N}_2^{\alpha,i}\cdot\mathbf{V}_{2,2}^{\alpha,i}}{|\mathbf{N}_2^{\alpha,i}|\,|\mathbf{V}_{2,2}^{\alpha,i}|}+\frac{\mathbf{N}_2^{\alpha,i}\cdot\mathbf{V}_{2,2}^{\alpha,i+1}}{|\mathbf{N}_2^{\alpha,i}|\,|\mathbf{V}_{2,2}^{\alpha,i+1}|}=f_{2,2,2}^{D}(x_2^{\alpha,i-1},x_2^{\alpha,i},x_2^{\alpha,i+1})=f_{2,2,2}^{D}(a,b,c)$$

$$=\left\{\frac{2ab-a^2-b^2}{[(b-a)^2+(b^2-a^2)^2]^{\frac{1}{2}}}+\frac{b^2-2bc+c^2}{[(c-b)^2+(c^2-b^2)^2]^{\frac{1}{2}}}\right\}(1+4b^2)^{-\frac{1}{2}},$$

$$\frac{\mathbf{N}_2^{\alpha,1}\cdot\mathbf{V}_{0,2}^{\alpha,1}}{|\mathbf{N}_2^{\alpha,1}|\,|\mathbf{V}_{0,2}^{\alpha,1}|}+\frac{\mathbf{N}_2^{\alpha,1}\cdot\mathbf{V}_{2,2}^{\alpha,2}}{|\mathbf{N}_2^{\alpha,1}|\,|\mathbf{V}_{2,2}^{\alpha,2}|}=f_{0,2,2}^{D}(y_0^{\alpha,0},x_2^{\alpha,1},x_2^{\alpha,2})=f_{0,2,2}^{D}(a,b,c)$$

$$=\left\{\frac{2b-b^2-a-1}{[(b-1)^2+(b^2-a-1)^2]^{\frac{1}{2}}}+\frac{b^2-2bc+c^2}{[(c-b)^2+(c^2-b^2)^2]^{\frac{1}{2}}}\right\}(1+4b^2)^{-\frac{1}{2}},$$

$$\frac{\mathbf{N}_2^{\alpha,i}\cdot\mathbf{V}_{1,2}^{\alpha,i}}{|\mathbf{N}_2^{\alpha,i}|\,|\mathbf{V}_{1,2}^{\alpha,i}|}+\frac{\mathbf{N}_2^{\alpha,i}\cdot\mathbf{V}_{2,3}^{\alpha,i+1}}{|\mathbf{N}_2^{\alpha,i}|\,|\mathbf{V}_{2,3}^{\alpha,i+1}|}=f_{1,2,3}^{D}(x_1^{\alpha,i-1},x_2^{\alpha,i},y_3^{\alpha,i+1})=f_{1,2,3}^{D}(a,b,c)$$

$$=\left\{\frac{2ab-b^2-a^2-1}{[(b-a)^2+(b^2-a^2-1)^2]^{\frac{1}{2}}}+\frac{b^2+2b+c+1}{[(1+b)^2+(c-b^2+1)^2]^{\frac{1}{2}}}\right\}(1+4b^2)^{-\frac{1}{2}},$$

$$\frac{\mathbf{N}_2^{\alpha,i}\cdot\mathbf{V}_{2,2}^{\alpha,i}}{|\mathbf{N}_2^{\alpha,i}|\,|\mathbf{V}_{2,2}^{\alpha,i}|}+\frac{\mathbf{N}_2^{\alpha,i}\cdot\mathbf{V}_{2,3}^{\alpha,i+1}}{|\mathbf{N}_2^{\alpha,i}|\,|\mathbf{V}_{2,3}^{\alpha,i+1}|}=f_{2,2,3}^{D}(x_2^{\alpha,i-1},x_2^{\alpha,i},y_3^{\alpha,i+1})=f_{2,2,3}^{D}(a,b,c)$$

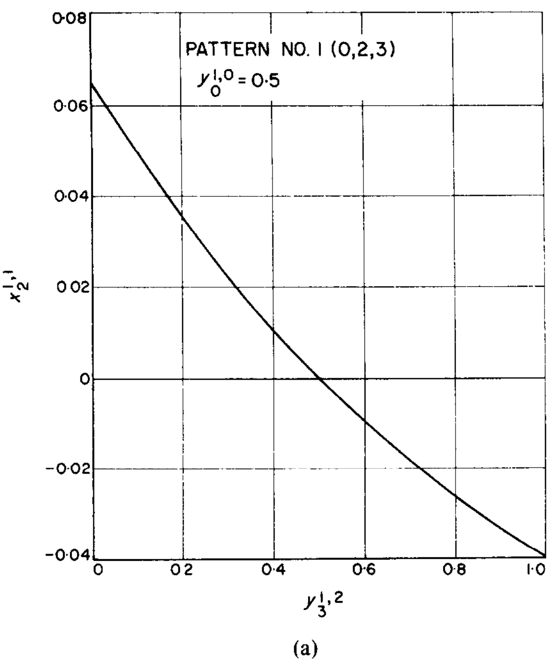
$$=\left\{\frac{2ab-a^2-b^2}{[(b-a)^2-(b^2-a^2)^2]^{\frac{1}{2}}}+\frac{b^2+2b+c+1}{[(1+b)^2+(c-b^2+1)^2]^{\frac{1}{2}}}\right\}(1+4b^2)^{-\frac{1}{2}},$$

$$\frac{\mathbf{N}_1^{\alpha,i}\cdot\mathbf{V}_{2,1}^{\alpha,i}}{|\mathbf{N}_1^{\alpha,i}|\,|\mathbf{V}_{2,1}^{\alpha,i}|}+\frac{\mathbf{N}_1^{\alpha,i}\cdot\mathbf{V}_{1,3}^{\alpha,i+1}}{|\mathbf{N}_1^{\alpha,i}|\,|\mathbf{V}_{1,3}^{\alpha,i+1}|}=f_{2,1,3}^{D}(x_2^{\alpha-1},x_1^{\alpha,i},y_3^{\alpha,i+1})=f_{2,1,3}^{D}(a,b,c)$$

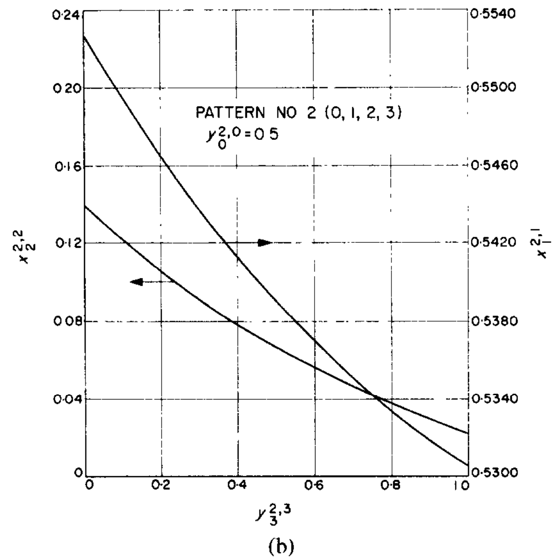
$$=\left\{\frac{b^2-2ab+a^2-1}{[(b-a)^2+(b^2-a^2+1)^2]^{\frac{1}{2}}}+\frac{b^2+2b+c}{[(1+b)^2+(c-b^2)^2]^{\frac{1}{2}}}\right\}(1+4b^2)^{-\frac{1}{2}}.$$

Note that by reciprocity:

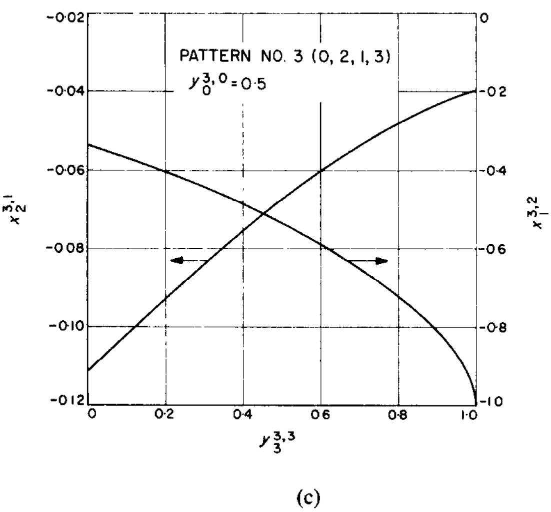
$$f_{1,2,2}^{D}(x_1^{\alpha,i-1},x_2^{\alpha,i},x_2^{\alpha,i+1})=-f_{2,2,1}^{D}(x_2^{\alpha,i-1},x_2^{\alpha,i},x_1^{\alpha,i+1}).$$



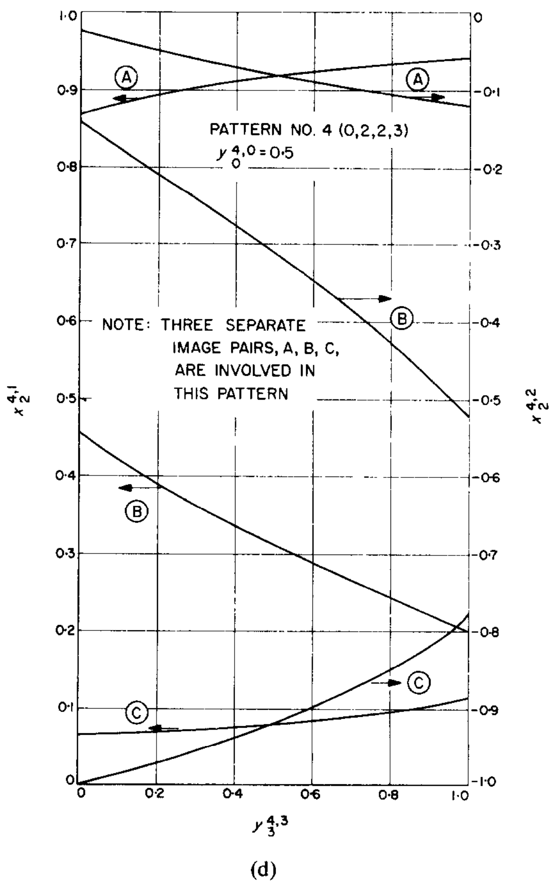
(a)



(b)



(c)



(d)

FIG. 6. Relationship between coordinates of source, reflection, and receiver points for patterns No. 1 through No. 4.

The system of simultaneous equations to be solved for the first few patterns is:

pattern No. 1 (0, 2, 3)

$$f_{0,2,3}(y_0^{1,0}, x_2^{1,1}, y_3^{1,2}) = 0,$$

pattern No. 2 (0, 1, 2, 3)

$$f_{0,1,2}(y_0^{2,0}, x_1^{2,1}, x_2^{2,2}) = 0,$$

pattern No. 3 (0, 2, 1, 3)

$$f_{0,2,1}(y_0^{3,0}, x_2^{3,1}, x_1^{3,2}) = 0,$$

$$f_{2,1,3}(x_2^{3,1}, x_1^{3,2}, y_3^{3,3}) = 0,$$

pattern No. 4 (0, 2, 2, 3)

$$f_{0,2,2}(y_0^{4,0}, x_2^{4,1}, x_2^{4,2}) = 0,$$

$$f_{2,2,3}(x_2^{4,1}, x_2^{4,2}, y_3^{4,3}) = 0,$$

pattern No. 5 (0, 1, 2, 2, 3)

$$f_{0,1,2}(y_0^{5,0}, x_1^{5,1}, x_2^{5,2}) = 0,$$

$$f_{1,2,2}(x_1^{5,1}, x_2^{5,2}, x_2^{5,3}) = 0,$$

$$f_{2,2,3}(x_2^{5,2}, x_2^{5,3}, y_3^{5,4}) = 0,$$

pattern No. 6 (0, 2, 2, 2, 3)

$$f_{0,2,2}(y_0^{6,0}, x_2^{6,1}, x_2^{6,2}) = 0,$$

$$f_{2,2,2}(x_2^{6,1}, x_2^{6,2}, x_2^{6,3}) = 0,$$

$$f_{2,2,3}(x_2^{6,2}, x_2^{6,3}, y_3^{6,4}) = 0,$$

pattern No. 7 (0, 2, 2, 1, 3)

$$f_{0,2,2}(y_0^{7,0}, x_2^{7,1}, x_2^{7,2}) = 0,$$

$$f_{2,2,1}(x_2^{7,1}, x_2^{7,2}, x_1^{7,3}) = 0,$$

$$f_{2,1,3}(x_2^{7,2}, x_1^{7,3}, y_3^{7,4}) = 0,$$

pattern No. 8 (0, 2, 1, 2, 3)

$$f_{0,2,1}(y_0^{8,0}, x_2^{8,1}, x_1^{8,2}) = 0,$$

$$f_{2,1,2}(x_2^{8,1}, x_1^{8,2}, x_2^{8,3}) = 0,$$

$$f_{1,2,3}(x_1^{8,2}, x_2^{8,3}, y_3^{8,4}) = 0,$$

pattern No. 9 (0, 1, 2, 1, 3)

$$f_{0,1,2}(y_0^{9,0}, x_1^{9,1}, x_2^{9,2}) = 0,$$

$$f_{1,2,1}(x_1^{9,1}, x_2^{9,2}, x_1^{9,3}) = 0,$$

$$f_{2,1,3}(x_2^{9,2}, x_1^{9,3}, y_3^{9,4}) = 0.$$

The relationship between the coordinates of the first four patterns is shown graphically in Fig. 6. It is interesting to note that pattern No. 4 gives rise to three secondary images.

4. CONCLUSION

Although the above approach allows a straightforward formulation of the system of equations describing the multiple reflection patterns associated with a configuration containing curved specular surfaces with a minimum of geometrical analyses and the computation procedure is reduced to a familiar form, the solution of the system of equations and the computation of the view function and view factor remain a formidable task. However, the storage capacity and speed of computation of the modern electronic digital computer and the techniques of numerical analysis are equal to the challenge. These, then, would indicate the feasibility of a view factor program for generalized curved specular surface configuration.

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Résumé—On présente des expressions mathématiques générales pour (1) le facteur de forme correspondant seulement à l'image primaire d'une surface vue par une autre surface dans un réflecteur spéculaire non-plan, et (2) le facteur de forme pour l'image secondaire dans une situation d'images multiples faisant intervenir des réflecteurs spéculaires non-plans. La notation vectorielle est employée pour présenter les formules générales qui sont indépendantes d'un système spécifique de coordonnées. Les expressions résultantes sont simples dans leur concept et sont identifiables avec l'expression connue communément pour le facteur de forme entre des surfaces diffuses. Les formules étendent l'application de la "méthode des images" à des enceintes contenant des surfaces spéculaires de forme arbitraire. On donne deux exemples qui illustrent l'emploi des expressions générales.

Zusammenfassung—Allgemeine mathematiche Ausdrücke werden angegeben für (1) den Anordnungsfaktor nur für die Primiärabbildung einer Oberfläche gesehen von einer zweiten Oberfläche aus in einem nichtebenen spiegelnden Reflektor und (2) den Anordnungsfaktor für die sekundäre Abbildung im vielfach abbildenden Fall bei nichtebenen spiegelnden Reflektoren. Die Vektorbezeichnung wird verwendet, um die allgemeinen Formulierungen wiederzugeben, die unabhängig von einem spezifischen Koordinatensystem sind. Die resultierenden Ausdrücke sind begrifflich einfach und sind vergleichbar mit den allgemein bekannten Ausdrücken für den Anordnungsfaktor zwischen diffus strahlenden Oberflächen. Die Formulierungen erweitern die Anwendung der "Abbildungsmethode" auf Verhältnisse mit spiegelnden Oberflächen mit beliebiger Topographie. Zwei Beispiele sind angegeben, um die Anwendung der allgemeinen Ausdrücke zu erläutern.

Аннотация—Приведены общие математические выражения, во-первых, для углового коэффициента только для первичного изображения одной поверхности с точки зрения другой поверхности в неплоском зеркальном отражателе, и во-вторых, углового коэффициента в случае множественных изображений в неплоском зеркальном отражателе. Общие формулировки даются в векторном выражении и не зависят от системы координат. Полученные выражения очень просты и могут быть легко сведены к известным выражениям для углового коэффициента между рассеивающими поверхностями. Эти формулировки позволяют применить метод «изображений» для оболочек отражающих поверхностей произвольной топографии. Для иллюстрации общих выражений приводятся два примера.